Mansour's Conjecture is True for Random DNF Formulas

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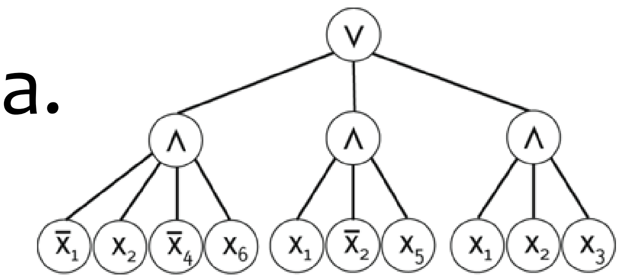
[M94]

A Conjecture

Let f be a t-term DNF formula.



∃ a real $t^{O(\log 1/\varepsilon)}$-term poly p s.t.

$^{3}/_{16}x_1x_6 - {}^{1}/_{8}x_3x_7x_8$

$$\mathop{\mathrm{E}}_{x\in\{0,1\}^n}[(p(x)-f(x))^2] \le \varepsilon$$

[KM91]

Sparse Approximators

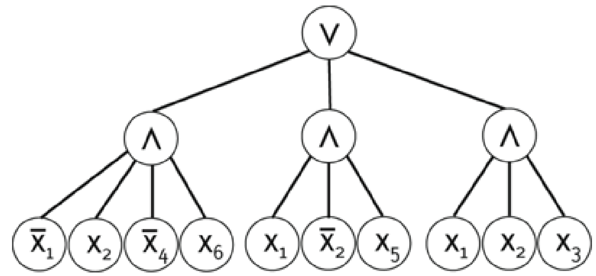
Thm: If $\forall f \in C$ has an s-sparse ε-approx p, then there is a uniform distribution MQ PAC learner for C that runs in time poly(n,s,ε^{-1}).

MC $\Rightarrow$ PAC-learning DNF

[J94]

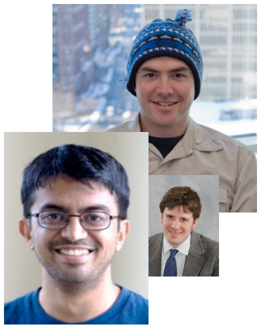
The Harmonic Sieve

A MQ PAC-learner for poly(n)-term DNF formulas over the uniform distribution.



- Didn’t prove MC.
- Used weak-approximator + boosting.

…and a decade passed.



[GKK08]

Sparse Approximators

Thm: If $\forall f \in C$ has an s-sparse ε-approx p, then there is a uniform distribution MQ *agnostic* learner for C that runs in time poly(n,s,ε^{-1}).

MC $\Rightarrow$ agnostic-learning DNF

Agnostic Learning

- f arbitrary Boolean function
- $\text{opt} = \min_{c \in C} \Pr_x[c(x) \neq f(x)]$

An <u>*agnostic learner*</u> is given MQ to f

w.h.p. outputs h s.t.

$$\Pr_x[h(x) \neq f(x)] \leq \text{opt} + \varepsilon$$

Previous Results

$$\mathop{E}_{x\in\{0,1\}^n}[(p(x)-f(x))^2] \leq \varepsilon$$

f a t-term DNF formula; $\exists\varepsilon$-approx p with:

- degree $O(\log(t/\varepsilon)^2)$ [LMN89]
- $t^{O(\log\log t \log(1/\varepsilon))}$ terms [M92]
- degree $O(\log(t/\varepsilon))$ [H01]

Our Results

$$\mathop{E}_{x\in\{0,1\}^n}[(p(x)-f(x))^2] \leq \varepsilon$$

$\exists$ ε-approx p with $t^{O(\log(1/\varepsilon))}$ terms for

- f a t-term *random* DNF formula
- f a t-term *read-k* DNF formula

(and [GK08] gives agnostic learners)

Outline

1. Intro
2. How we didn't prove it.
3. How we did prove it.
   a) Read-once DNF formulas
   b) Random DNF formulas
   c) Read-k DNF formulas
4. Pseudorandomness

How we didn't prove Mansour's Conjecture 1

Every f has a unique real polynomial representation with coeffs $\hat{f}(S)$ (the Fourier representation).

Analyze the large coeffs using Håstad's random restriction machinery [LMN89,M92,H01].

How we didn't prove Mansour's Conjecture 2

Entropy-Influence Conjecture: $E(f)=O(I(f))$

$\sum_S \hat{f}(S)^2 = 1$

$E(f) := \sum_S -\hat{f}(S)^2 \log(\hat{f}(S)^2)$

$I(f) := \sum_S |S| \hat{f}(S)^2$

$EI \Rightarrow MC$

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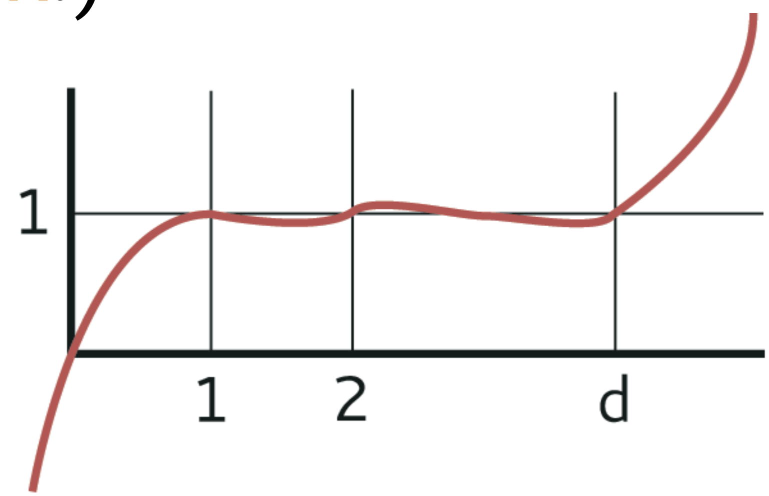
Polynomial Interpolation

$f = T_1 \vee T_2 \vee \cdots \vee T_t$

Let $y_f(x) = T_1 + T_2 + \cdots + T_t$
(# of terms satisfied by x.)

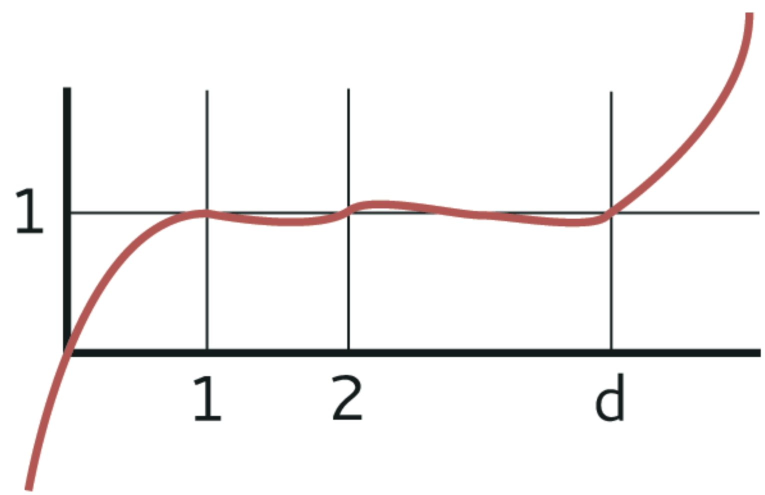
Interpolate the values of f on $\{x : y_f(x) \leq d\}$



The Polynomial

$$P_d(y) = ((-1)^{d+1}/d!)(y-1)(y-2)\cdots(y-d) + 1$$

- $P_d(0)=0$
- $P_d(y)=1,\ y=1\ldots d$
- $|P_d(y)|<\binom{y}{d},\ y>d$



The Polynomial

$$P_d(y) = ((-1)^{d+1}/d!)(y-1)(y-2)\cdots(y-d) + 1$$

- $P_d(y_f(x))$ has $t^{O(d)}$ terms.
- $P_d(y_f(x))=f(x)$ when x satisfies at most d terms.
- Need to show that x satisfies more terms with small probability.

Read-once DNF Formulas

Read-once: each var appears at most once

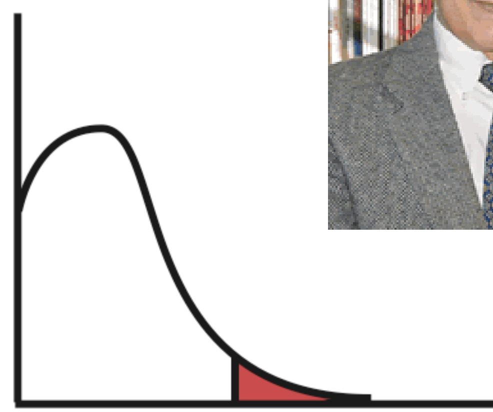
$x_1\bar{x}_5x_8 \vee x_2x_3\bar{x}_{18}x_{31} \vee x_4x_7$

$\Rightarrow$ terms are satisfied independently.

How do we show that sums of independent variables are concentrated in a narrow range?

Chernoff Bounds

- $T = \sum_{i=1}^{t} T_i$ (i.r.v.'s T_i=1 w.p. μ_i)
- $\mu = \sum_{i=1}^{t} \mu_i = E[T]$
- Can assume
  $\mu \leq \log(1/\varepsilon)$, or $f \approx 1$.



Chernoff : $\Pr[\, T = j \,] \leq (e\mu/j)^j$

MC is true for RO DNFs

$$\mathop{E}_{x \in \{0,1\}^n}[(p(x)-f(x))^2] \leq \varepsilon$$

$$\sum_{j=0}^{t} \Pr[y_f(x)=j]\,(P_d(y_f(x))-f(x))^2$$

$$\leq \sum_{j=d+1}^{t} (ed/j)^j \binom{j}{d}^2 \leq \varepsilon$$

for $d = \log(1/\varepsilon)$.

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MC is true for random DNFs

Our model: choose each term of a t-term DNF from the set of all terms of length $\log(t)$.

Show that w.h.p. random DNFs behave like RO DNFs using the method of bouded differences.

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Read-k DNF Formulas

Read-k: each var appears at most k times

$x_1\overline{x}_5x_8 \vee x_1x_2\overline{x}_3x_4 \vee x_5x_7$

Terms are no longer independent!

The Modified Construction

$f = T_1 \vee T_2 \vee \cdots \vee T_t$ (ordered from longest to shortest)

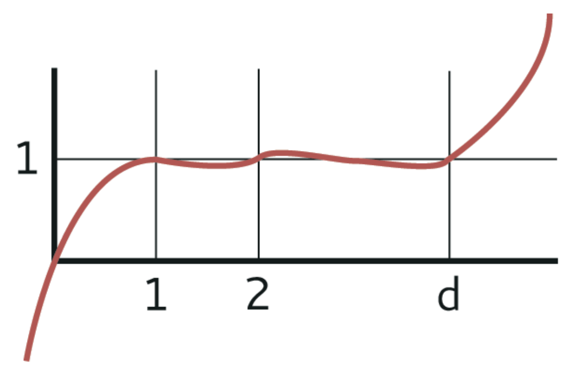
Let $z_f(x) = A_1 + A_2 + \cdots A_t$

$A_i = T_i \wedge (\wedge_{j \sim i, j \leq i} \neg T_j)$

(# of ind. terms sat. by x)

Interpolate the values

of f on $\{x : z_f(x) \leq d\}$



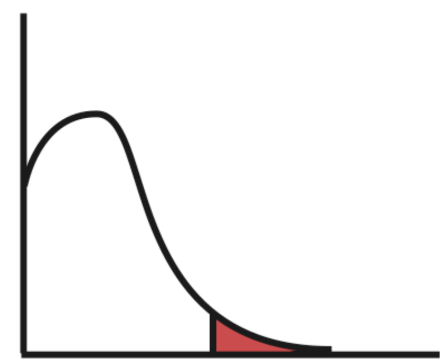
The Polynomial

$$P_d(z) = ((-1)^{d+1}/d!)(z-1)\,(z-2)\cdots(z-d) + 1$$

- $P_d(z_f(x))$ has $t^{O(kd)}$ terms.
- $P_d(z_f(x))=f(x)$ when x satisfies $\leq d$ ind. terms.
- Need to show that x satisfies more indep. terms with small probability.

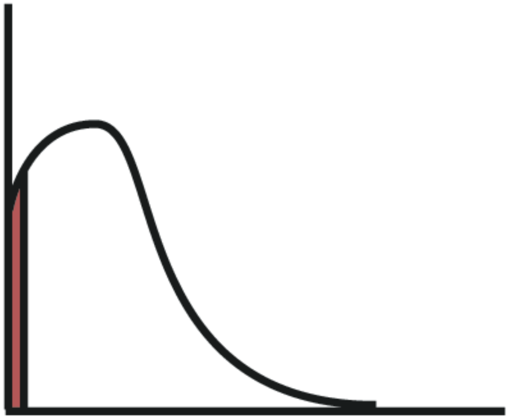
Concentration for Read-k

- T_i are r.v.'s 1 w.p. μ_i
- $\mu = \sum_{i=1}^{t} \mu_i$
- $A = \sum_{i=1}^{t} A_i \quad (A_i = T_i \wedge (\wedge_{j \sim i, j \leq i} \neg T_j))$
- $\Pr[\ A = j\ ] \leq \sum_{|S|=j} \Pi_{i \in S} T_i \leq (e\mu/j)^j$



Janson Bounds

- T_i are r.v.'s 1 w.p. μ_i
- $\mu = \sum_{i=1}^{t} \mu_i$
- $\Delta = \sum_{i \sim j} E[\, T_i T_j \,]$
- $\Pr[\, T=0 \,] \leq \exp(-\mu^2/\Delta)$



By Janson, can assume

$\mu \leq 16^k \log(1/\varepsilon)$, or $f \approx 1$.

Recap

$$\underset{x \in \{0,1\}^n}{E}[(p(x)-f(x))^2] \leq \varepsilon$$

$\exists\,\varepsilon$-approx p with $t^{O(\log(1/\varepsilon))}$ terms for

f a t-term *random* DNF formula w.h.p.

$\exists\,\varepsilon$-approx p with $t^{O(16^k \log(1/\varepsilon))}$ terms for

f a t-term *read-k* DNF formula

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Pseudorandomness

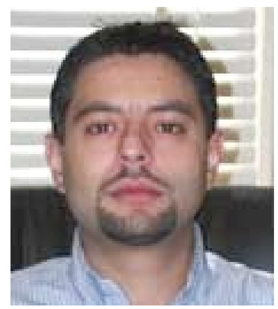
A distribution X ϕ-fools C if $\forall f \in C$

$$|E[f(X)] - E[f(U)]| \leq \phi$$

Seed length is # of random bits used by X.

PRGs against DNFs

Seed length for pseudorandom generators against t-term DNF formulas:



- $O(\log^4(tn/\phi))$ [LVW93]
- $O(\log(n)\log^2(t/\phi))$ [B07]
- $O(\log(n) + \log^2(t/\phi)\log\log(t/\phi))$ [DETT10]

The Sandwich Bound

If $\exists$ $s(\phi)$-sparse g & h s.t.

$\forall x,\ g(x) \leq f(x) \leq h(x)$

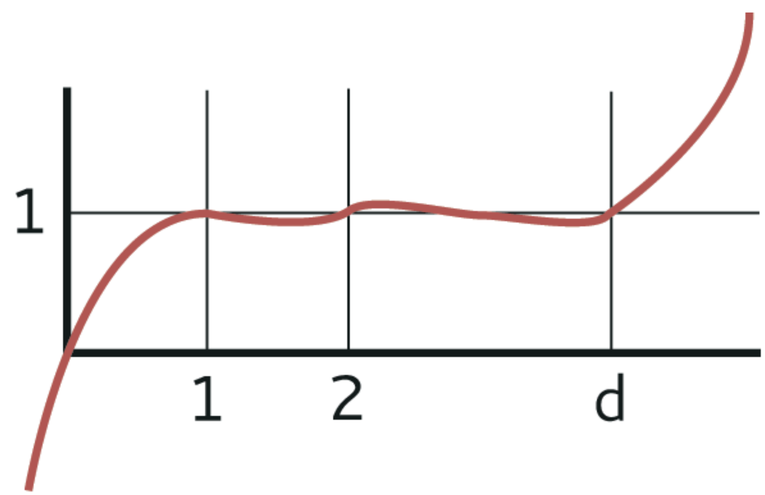
$E[h(x) - f(x)] \leq \phi$, $\quad$ $E[f(x) - g(x)] \leq \phi$

Then $\exists$ dist. that ϕ-fools f with seed length $O(\log n + \log s(\phi))$ [B07,DETT10]

The Polynomial

$$P_d(y) = ((-1)^{d+1}/d!)(y-1)(y-2)\cdots(y-d) + 1$$

- $P_d(0)=0$
- $P_d(y)=1,\ y=1\ldots d$
- $|P_d(y)|<\binom{y}{d}$
- $P_d(y)>1,\ y>d,\ d$ odd
- $P_d(y)<0,\ y>d,\ d$ even



PRGs against DNFs

- t-term random DNFs are fooled by PRGs w/ seed length
  $O(\log(n) + \log(t)\log(1/\phi))$ w.h.p.

- t-term read-k DNFs are fooled by PRGs w/ seed length
  $O(\log(n) + \log(t)16^k\log(1/\phi))$

([DETT10] showed $O(\log(n) + \log(t)\log(1/\phi))$ for RO DNFs)

Open Problems

- Prove Mansour's Conjecture for all t-term DNF formulas.

- Show PRGs against DNFs with seed length $O(\log(t)\log(1/\phi))$.

The End